

A GENERIC IDENTIFICATION THEOREM FOR GROUPS OF FINITE MORLEY RANK, REVISITED

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ABSTRACT. This paper contains a stronger version of a final identification theorem for the ‘generic’ groups of finite Morley rank.

1. INTRODUCTION

This paper contains a version of a generic identification theorem for simple groups of finite Morley rank adapted from the main result of [5] by weakening its assumptions. It is published because it is being used in the authors’ study of actions of groups of finite Morley rank [6].

A general discussion of the subject can be found in the books [7] and [3].

A group of finite Morley rank is said to be of p' -type, if it contains no infinite abelian subgroup of exponent p . Notice that a simple algebraic group over an algebraically closed field K is of p' -type if and only if $\text{char } K \neq p$. Other definitions can be found in the next section.

The aim of this work is to prove the following.

Theorem 1.1. *Let G be a simple group of finite Morley rank and D a maximal p -torus in G of Prüfer rank at least 3. Assume that*

- (A) *Every proper connected definable subgroup of G which contains D is a K -group.*
- (B) *For every element x of order p in D , the group $C_G^\circ(x)$ is of p' -type and $C_G^\circ(x) = F^\circ(C_G^\circ(x))E(C_G^\circ(x))$.*
- (C) *$\langle C_G^\circ(x) \mid x \in D, |x| = p \rangle = G$.*

Then G is a Chevalley group over an algebraically closed field of characteristic distinct from p .

Notice that, under assumption (B) of Theorem 1.1, $C_G^\circ(x)$ is a central product of $F^\circ(C_G^\circ(x))$ and $E(C_G^\circ(x))$.

The predecessor of this theorem, Theorem 1.2 of [5], was based on a stronger assumption than our assumption (A), namely, that *every* proper definable subgroup of G is a K -group, a condition that was difficult to check in its actual applications.

The proof of Theorem 1.1 is given in Section 3.

1.1. Definitions. All definitions related to groups of finite Morley rank in general can be found in [7] and [3].

From now on G is a group of finite Morley rank. The group G is called a K -group, if every infinite simple definable and connected section of the group is an algebraic group over an algebraically closed field.

A p -torus S is a direct product of finitely many copies of the quasicyclic group \mathbb{Z}_{p^∞} . The number of copies is called the *Prüfer p -rank* of S and is denoted by $\text{pr}(S)$.

For a definable group H , $\text{pr}(H)$ is the maximum of the Prüfer ranks of p -subgroups in H . It is easy to see that the Prüfer p -rank of any subgroup of a group of finite Morley rank is finite.

A group H is called *quasi-simple* if $H' = H$ and $H/Z(H)$ is simple and non-abelian. A quasi-simple subnormal subgroup of G is called a *component* of G . The product of all components of G is called the *layer* of G and denoted by $L(G)$, and $E(G)$ stands for $L^\circ(G)$. It is known (see Lemmas 7.6 and 7.10 in [7]) that G has finitely many components and that they are definable and are normal in $E(G)$.

$F(H)$ is the *Fitting subgroup* of H , that is, the maximal normal definable nilpotent subgroup.

If H is a group of finite Morley rank then $B(H)$ is the subgroup generated by all definable connected 2-subgroups of bounded exponent in H . Note that $B(H)$ is connected by Assertion 2.3.

2. BACKGROUND MATERIAL

2.1. Algebraic Groups. For a discussion of the model theory of algebraic groups, the reader might like to see Section 3.1 in [4]. The basic structural facts and definitions related to algebraic groups can be easily found in the standard references such as [8, 11].

First note that a connected algebraic group G is called *simple* if it has no proper normal connected and closed subgroups. Such a group turns out to have a finite center, the quotient group being simple as an abstract group. The classical classification theorem for simple algebraic groups states that simple algebraic groups over algebraically closed fields are Chevalley groups, that is, groups constructed from Chevalley bases in simple complex Lie algebras as described, for example, in [8].

Now fix a maximal torus T in a connected algebraic group G and denote the corresponding root system by Φ , and for each $\alpha \in \Phi$, denote the corresponding root subgroup by X_α . The subgroup $\langle X_\alpha, X_{-\alpha} \rangle$ is known to be isomorphic to SL_2 or PSL_2 and is called a *root SL_2 -subgroup*.

If G is simple, the roots can have at most two different lengths, and the terms ‘short root SL_2 -subgroup’ and ‘long root SL_2 -subgroup’ have the obvious meanings.

A simple algebraic group is generated by its root SL_2 -subgroups. In a simple algebraic group, all long root SL_2 -subgroups are conjugate to each other, and similarly all short root SL_2 -subgroups are conjugate to each other.

Assertion 2.1. *Suppose that G is a simple algebraic group over an algebraically closed field. Let T be a maximal torus in G and K, L closed subgroups of G that are isomorphic to SL_2 or PSL_2 and are normalised by T . Then the following hold.*

- (1) *Either $[K, L] = 1$ or $\langle K, L \rangle$ is a simple algebraic group of rank 2; that is of type A_2, B_2 or G_2 .*
- (2) *The subgroups K and L are embedded in $\langle K, L \rangle$ as root SL_2 -subgroups.*
- (3) *If $\langle K, L \rangle$ is of type G_2 , then $G = \langle K, L \rangle$.*

Proof. The proof follows from the description of closed subgroups in simple algebraic groups normalised by a maximal torus [12, 2.5]; see also [13, Section 3.1]. \square

Assertion 2.2. *Let G be a simple algebraic group over an algebraically closed field of characteristic $\neq p$, and let D be a maximal p -torus in G . Then $C_G(D)$ is a maximal torus in G .*

Proof. The proof follows from the description of centralisers of subgroups of commuting semisimple elements in simple algebraic groups [15, Theorem 5.5.8].

2.2. Groups of Finite Morley Rank.

Assertion 2.3. (Zil'ber's Indecomposability Theorem) *A subgroup of a group of finite Morley rank which is generated by a family of definable connected subgroups is also definable and connected.*

Proof. See [16] or Corollary 5.28 in [7].

Assertion 2.4. [7, Theorem 8.4] *Let $G \rtimes H$ be a group of finite Morley rank, where G and H are definable, G is an infinite simple algebraic group over an algebraically closed field and $C_H(G) = 1$. Then H can be viewed as a subgroup of the group of automorphisms of G , and moreover H lies in the product of the group of inner automorphisms and the group of graph automorphisms of G (which preserve root lengths). In particular, when H is connected, then H consists of inner automorphisms only.*

Assertion 2.5. [1] *Suppose G is a group of finite Morley rank, $G = G'$, and $G/Z(G)$ is a simple algebraic group over an algebraically closed field, and is of finite Morley rank, then $Z(G)$ is finite and G is also algebraic.*

Lemma 2.6. *Let G be a connected K -group of p' -type and D a maximal p -torus in G . If $L \triangleleft G$ is a component in G , then $D \cap L$ is a maximal p -torus in L and $D = C_D(L)(D \cap L)$.*

Proof. As G is connected, $L \triangleleft G$. Now the lemma immediately follows from Assertions 2.5, 2.4 and 2.2.

Lemma 2.7. *Under the assumptions of Theorem 1.1, we have, for every p -element $t \in D$,*

$$C_G^\circ(t) = F \cdot L_1 \cdots L_r,$$

where $F = F^\circ(C_G^\circ(t))$ and each L_i is a simple algebraic group over an algebraically closed field of characteristic $\neq p$.

Proof. For every p -element t in G , $C_G^\circ(t) = F \cdot E(C_G^\circ(t))$. And assumption (C) ensures that $C_G^\circ(t)$ is a K -group, hence its components are algebraic groups by Assertion 2.5.

2.3. Lyons's Theorem. A detailed discussion of this particular version of Lyons's theorem can be found in [4].

Assertion 2.8 ((Lyons [9, 10])). *Suppose that \mathbb{F} is an algebraically closed field, I is one of the connected Dynkin diagrams of the simple algebraic groups of Tits rank at least 3 and \tilde{G} is the simply connected simple algebraic group of type I over \mathbb{F} . Let G be an arbitrary group and for each $i \in I$, K_i stand for a subgroup of G which is centrally isomorphic to $PSL_2(\mathbb{F})$, and $T_i < K_i$ denote a maximal torus in K_i . Also assume that the following statements hold.*

- (1) *The group G is generated by K_i where $i \in I$.*
- (2) *For all $i, j \in I$, $[T_i, T_j] = 1$.*
- (3) *If $i \neq j$ and (i, j) is not an edge in I , then $[K_i, K_j] = 1$.*
- (4) *If (i, j) is a single edge in I , then $G_{ij} = \langle K_i, K_j \rangle$ is isomorphic to $(P)SL_3(\mathbb{F})$.*

- (5) If (i, j) is a double edge in I , then $G_{ij} = \langle K_i, K_j \rangle$ is isomorphic to $(P)Sp_4(\mathbb{F})$. Moreover, in that case, if $r_i \in N_{K_i}(T_i T_j)$ and $r_j \in N_{K_j}(T_i T_j)$ are involutions, then the order of $r_i r_j$ in $N_{G_{ij}}(T_i T_j)/T_i T_j$ is 4.
- (6) For all $i, j \in I$, K_i and K_j are root SL_2 -subgroups of G_{ij} corresponding to the maximal torus $T_i T_j$ of G_{ij} .

Then there is a homomorphism from \tilde{G} onto G , under which the root SL_2 -subgroups of \tilde{G} (for some simple root system) correspond to the subgroups K_i .

2.4. Reflection Groups. A linear semisimple transformation of finite order is called a *reflection* if it has exactly one eigenvalue which is not 1.

Theorem 2.9. *Let W be a finite group and assume that the following statements hold.*

- (1) *There is a normal subset $S \subseteq W$ consisting of involutions and generating W .*
- (2) *Over \mathbb{C} , W has a faithful irreducible representation of dimension $n \geq 3$ in which involutions from S act as reflections.*
- (3) *For almost all prime numbers q , W has faithful irreducible representations (possibly of different dimensions) over fields \mathbb{F}_q . Moreover, for every such representation, involutions in S act as reflections.*

Then W is one of the groups $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$ for $n \geq 3$.

Proof. A proof, based on the classification of irreducible complex reflection groups [14], can be found in [4].

3. PROOF OF THEOREM 1.1

The strategy is to construct the Weyl group and the root system of G , and then to apply Lyons's Theorem. So from now on G is a simple group of finite Morley rank and D is a maximal p -torus in G of Prüfer rank ≥ 3 and such that every proper definable connected subgroup containing D is a K -group. We also assume that $C_G^\circ(x)$ is of p' -type for every element $x \in D$ of order p , $C_G^\circ(x) = F^\circ(C_G^\circ(x))E(C_G^\circ(x))$ and

$$G = \langle C_G^\circ(x) \mid x \in D, |x| = p \rangle.$$

Notice that if M is a proper definable subgroup in G normalised by D then MD is a proper definable subgroup of G , for otherwise M would be normal in G , which contradicts simplicity of G . Therefore, MD and hence M are K -groups by assumption (A). This observation will be used throughout the proof.

We also shall systematically use the following observation.

Lemma 3.1. *$F^\circ(C_G^\circ(x))$ centralises D for every element $x \in D$ of order p .*

Proof. The result immediately follows from the fact that a p -torus in a definable nilpotent group belongs to the center of this group [7, Theorem 6.9].

3.1. Root Subgroups. From now on, SL_2 will be used instead of $SL_2(\mathbb{F})$, etc. Denote by Σ the set of all definable subgroups isomorphic to $(P)SL_2$ and normalised by D . These are our future root SL_2 -subgroups. If N is a subgroup of G which is normalised by D , then set $H_N := C_N(D \cap N)$. Note that if $K \in \Sigma$, then H_K is an algebraic torus in K .

Lemma 3.2. *The set Σ is non-empty.*

Proof. Assume the contrary. If $L = E(C_G^\circ(x)) \neq 1$ for some element x of order p from D , then, L being a central product of simple algebraic groups, contains a definable SL_2 -subgroup normalised by D . Therefore $C_G^\circ(x) = F^\circ(C_G^\circ(x))$ centralises D by Lemma 3.1. But then

$$G = \langle C_G^\circ(x) \mid x \in D, |x| = p \rangle$$

centralises D which contradicts the assumption that G is simple.

Lemma 3.3. *Let $K, L \in \Sigma$ be distinct and set $M = \langle K, L \rangle$. Then the following statements hold.*

- (1) *The subgroup $C_D(K) \cap C_D(L) \neq 1$ and M is a K -group.*
- (2) *Either K and L commute or M is an algebraic group of type A_2 , B_2 or G_2 .*
- (3) *$D \cap M = (D \cap K)(D \cap L)$ is a maximal p -torus in M .*
- (4) *If K and L do not commute then H_M is a maximal algebraic torus of the algebraic group M , and K and L are root SL_2 -subgroups of the algebraic group M with respect to the maximal torus H_M .*
- (5) *For all $K, L \in \Sigma$, we have $[H_K, H_L] = 1$.*
- (6) *For any $K, L \in \Sigma$, if the p -tori $D \cap K$ and $D \cap L$ have intersection of order > 2 , then $K = L$.*

Proof. For the proof of $C_D(K) \cap C_D(L) \neq 1$ we refer the reader to the proof of [4, Lemma 9.3]. After that $M \leq C_G(C_D(K) \cap C_D(L))$ is a proper definable subgroup of G and is a K -group since D normalizes M .

(2)-(3) For $L \in \Sigma$ set $R_L = C_D^\circ(L)$. If $n = \text{pr}(D)$, then the Prüfer p -rank of R_L is $n - 1$. Note that since D is maximal and D centralises a p -torus in L , $D \cap L$ is a maximal p -torus in L . Now let x be an element of order p in $R_K \cap R_L$. Then $K, L \leq E(C_G(x))$ by the assumptions of the theorem. Set $E = E(C_G(x))$. It follows from Lemma 2.6 that the subgroup $D \cap E$ is a maximal p -torus of E , and the subgroups K and L , being D -invariant, lies in components of E . If K and L belong to different components of E , then they commute. Otherwise the component A that contains both K and L is a simple algebraic group, and moreover $D \cap A$ is a maximal p -torus in A . Hence the results follow from Assertion 2.1.

(4)-(6) These follow by inspecting case by case and Assertion 2.1.

Lemma 3.4. *The subgroups in Σ generate G .*

Proof. Let $x \in D$ be of order p , then by assumption (B) of the theorem

$$C_G^\circ(x) = F \cdot L_1 \cdots L_n,$$

where $F = F^\circ(C_G^\circ(x))$ and $L_i \triangleleft C_G^\circ(x)$ is a simple algebraic group, for each $i = 1, \dots, n$.

The first step is to prove that $L_1 \cdots L_n \leq \langle \Sigma \rangle$. Note that $D \leq C_G^\circ(x)$ and $D \cap L_i$ is a maximal p -torus in L_i by Lemma 2.6. Let H_i stand for the maximal algebraic torus in L_i containing $D \cap L_i$ and Γ_i be the collection of root SL_2 -subgroups in L_i normalised by H_i . Since $D \cap L_i \leq H_i$, $D \cap L_i$ normalises the subgroups in Γ_i . By Lemma 2.6, we have $D = C_D(L_i)(D \cap L_i)$, hence D normalises $\langle \Gamma_i \rangle = L_i$; that is $\Gamma_i \in \Sigma$ for each $i = 1, \dots, n$. This proves the first step.

Hence for each $x \in D$ of order p , $C_G^\circ(x) = F \cdot E(C_G^\circ(x)) \leq F \langle \Sigma \rangle \leq C_G(D) \langle \Sigma \rangle$. Therefore

$$G = \langle C_G^\circ(x) \mid x \in D, |x| = p \rangle \leq C_G(D) \langle \Sigma \rangle.$$

Since $C_G(D)$ normalises $\langle \Sigma \rangle$, we have $\langle \Sigma \rangle \trianglelefteq G$. Now the result follows, since G is simple.

We make Σ into a graph by taking SL_2 -subgroups $L \in \Sigma$ for vertices and connecting two vertices K and L by an edge if K and L do not commute.

Lemma 3.5. *The graph Σ is connected.*

Proof. Otherwise consider a decomposition $\Sigma = \Sigma' \cup \Sigma''$ of Σ into the union of two non-empty sets such that no vertex in Σ' is connected to a vertex in Σ'' . Then we have

$$G = \langle \Sigma \rangle = \langle \Sigma' \rangle \times \langle \Sigma'' \rangle,$$

which contradicts the assumption that G is simple.

Lemma 3.6. *If $L \in \Sigma$ then $L = E(C_G(C_D(L)))$.*

Proof. Let $\text{pr}(D) = n$, then $\text{pr}(C_G(C_D(L))) = n$ and $\text{pr}(E(C_G(C_D(L)))) = 1$. Since $L \leq E(C_G(C_D(L)))$ and $E(C_G(C_D(L)))$ is a central product of simple algebraic groups over algebraically closed fields of characteristic $\neq p$, we immediately conclude that $L = E(C_G(C_D(L)))$. \square

3.2. Weyl Group. Recall that when $L \in \Sigma$, H_L stands for the maximal algebraic torus $H_L := C_L(D \cap L)$ in $L \cong SL_2$. Now set $H = \langle H_L \mid L \in \Sigma \rangle$ and call it the *natural torus associated with D* . It easily follows from Lemma 3.3(5) that H is a divisible abelian group.

For any $L \in \Sigma$, $W(L) := N_L(H)H/H = N_L(H_L)/H_L$ is the Weyl group of SL_2 and has order 2; hence $W(L)$ contains a single involution, which will be denoted by r_L . Note that the subgroup L is uniquely determined by r_L , since $C_D(L) = C_D^\circ(r_L)$ and $L = E(C_G(C_D(L)))$ by Lemma 3.6.

Lemma 3.7. *Consider a graph Δ with the set of vertices Σ , in which two vertices K and L are connected by an edge if $[r_K, r_L] \neq 1$. If K and L belong to different connected components of Δ , then $[K, L] = 1$.*

Proof. It suffices to check this statement in the subgroup $M = \langle K, L \rangle$, where it is obvious by Lemma 3.3(2). \square

Notice that D is a p -torus, the subgroups $N_G(D)$ and $C_G(D)$ are definable and the factor group $N_G(D)/C_G(D)$ is finite. Set $W := N_G(D)/C_G(D)$. The images of involutions r_L , for $L \in \Sigma$, in W generate a subgroup which we denote by W_0 . Since, by their construction, involutions r_L normalise D , there is a natural action of W_0 on D .

Lemma 3.8. *The p -torus D lies in the natural torus H . In particular, D is the Sylow p -subgroup of H .*

Proof. Set $D' = \langle D \cap L \mid L \in \Sigma \rangle$. It suffices to prove that $D' = D$. First note that $D' \leq D \cap H$. If $D' < D$ then, since $[D, r_L] = D \cap L$, all involutions r_L act trivially on the factor group D/D' which is divisible. Let us take an element $d \in D$ which has sufficiently big order so that the image of $d^{|W_0|}$ in D/D' has order at least p^2 . Then the element

$$z = \prod_{w \in W_0} d^w$$

has the same image in D/D' as $d^{|W_0|}$ and thus z has order at least p^2 . Since $D = C_D(L)(D \cap L)$ and $|C_D(L) \cap (D \cap L)| \leq |Z(L)| \leq 2$, we see that $|C_D(r_L) : C_D(L)| \leq 2$. Of course, the equality is possible only if $p = 2$. In any case, since $z \in C_D(r_L)$, $z^p \in C_D(L)$ for all $L \in \Sigma$ and $z^p \neq 1$. Hence $z^p \in C_G(\langle \Sigma \rangle) = Z(G)$. This contradiction shows that $D = D'$ and $D \leq H$. \square

Lemma 3.9. $N_G(D) = N_G(H)$.

Proof. The embedding $N_G(H) \leq N_G(D)$ follows from Lemma 3.8. Vice versa, if $x \in N_G(D)$ then the action of x by conjugation leaves the set Σ invariant, hence it leaves invariant the set of algebraic tori $\{H_L \mid L \in \Sigma\}$ which generates H . Therefore $x \in N_G(H)$. \square

Lemma 3.10. $C_G(D) = C_G(H)$.

Proof. Let $x \in C_G(D)$, then, for every $L \in \Sigma$, x centralises $C_D(L)$ and thus, by Lemma 3.6, normalises $L = E(C_G(C_D(L)))$. Since x centralises a maximal p -torus $D \cap L$ of L , it centralises the maximal torus $H_L = C_L(D \cap L)$. Hence $x \in C_G(H)$. This proves $C_G(D) \leq C_G(H)$. The reverse inclusion follows from Lemma 3.8. \square

In view of Lemmata 3.9 and 3.10, we can refer to W_0 either as the subgroup generated by the images of involutions r_L in the factor group $N_G(D)/C_G(D)$ or as the subgroup generated by the images of involutions r_L in $N_G(H)/C_G(H)$. Also, we now know that the group W_0 acts on D faithfully.

3.3. Tate Module. Now the aim is to construct a \mathbb{Z} -lattice on which W_0 acts as a crystallographic reflection group. For that purpose we shall associate with D the Tate module T_p . It is constructed in the following way.

Let E_{p^k} be the subgroup of D generated by elements of order p^k . Notice that every r_L acts on E_{p^k} as a reflection, that is, $E_{p^k} = C_{E_{p^k}}(r_L) \times [E_{p^k}, r_L]$, $[E_{p^k}, r_L] \leq D \cap L$ is a cyclic group and r_L inverts every element in $[E_{p^k}, r_L]$.

Consider the sequence of subgroups

$$E_p \longleftarrow E_{p^2} \longleftarrow E_{p^3} \longleftarrow \cdots$$

linked by the homomorphisms $x \mapsto x^p$. The projective limit of this sequence is the free module T_p over the ring \mathbb{Z}_p of p -adic integers. The action of W_0 on D can be lifted to T_p , where it is still an irreducible reflection group. By construction, T_p/pT_p is isomorphic to E_p as a W_0 -module. Notice also that W_0 acts on the tensor product $T_p \otimes_{\mathbb{Z}_p} \mathbb{C}$ as a (complex) reflection group, and that the dimension of $T_p \otimes_{\mathbb{Z}_p} \mathbb{C}$ over \mathbb{C} coincides with the Prüfer p -rank of D , hence is at least 3.

3.4. More Reflection Representations for W_0 . Now let us focus on odd primes $q \neq p$. Consider the elementary abelian q -subgroups E_q generated in H by all elements of the fixed prime order q . For the sake of complete reducibility of the action of W on E_q , one can consider only $q > |W|$.

Lemmas 3.11, 3.15 and 3.16 below are similar to some results in [4]. We include the proofs here for the sake of completeness of exposition.

Lemma 3.11. [4, Lemma 9.7] *Let $N = N_G(H)$, then $C_N(E_q) = C_G(H)$.*

Proof. It is clear that $C_G(H) \leq C_N(E_q)$. To see the converse, let $x \in C_N(E_q)$. Since $x \in N$, it acts on the elements of Σ by conjugation.

First let us prove that x normalises each subgroup in Σ . To get a contradiction, assume that there is some subgroup $L \in \Sigma$ such that $L^x \neq L$. But then by Lemma 3.3, L and L^x either commute or generate a semisimple group as root SL_2 -subgroups. Hence $|L \cap L^x| \leq 2$. But this gives a contradiction since q is an odd prime and $L \cap E_q = L^x \cap E_q \leq L \cap L^x$.

Hence for each $L \in \Sigma$, $L^x = L$ and x acts on $H \cap L$ as an element from $N_L(H \cap L)$, since SL_2 does not have any definable outer automorphisms. Note that the Weyl group of SL_2 is generated by an involution which inverts the torus $H \cap L$. Since x centralises $E_q \cap H$, x centralises $H \cap L$ for each $L \in \Sigma$, and hence x centralises $H = \langle H \cap L \mid L \in \Sigma \rangle$ and $x \in C_G(H)$. This proves the equality. \square

Now notice that $[E_q, r_L]$ is generated by a q -element in H_L and thus has order q . Hence E_q is a finite dimensional vector space over \mathbb{F}_q on which W_0 acts as a group generated by reflections.

Lemma 3.12. *The group W_0 acts irreducibly on E_q .*

Proof. Note that W_0 acts on E_q faithfully by Lemma 3.11. Since $q > |W|$, the action of W_0 on E_q is completely reducible. So if the action is reducible, then we can write $E_q = E' \oplus E''$ for two proper W_0 -invariant subspaces.

Assume that W_0 acts trivially on one of the subspaces E' or E'' , say on E' . If $L \in \Sigma$, then $E_q = C_{E_q}(L) \times (E_q \cap L)$, and, obviously, $C_{E_q}(L) = C_{E_q}(r_L)$. Hence all $L \in \Sigma$ centralise E' and $E' \leq C_G(\langle \Sigma \rangle) = Z(G) = 1$. Therefore W_0 acts nontrivially on both E' and E'' .

For $L \in \Sigma$, the -1 -eigenspace $[E_q, r_L]$ of r_L belongs to one of the subspaces E' or E'' and hence r_L acts as a reflection on one of the subspaces E' or E'' and centralises the other. Set $\Sigma' = \{L \in \Sigma \mid [E_q, r_L] \leq E'\}$ and $\Sigma'' = \{L \in \Sigma \mid [E_q, r_L] \leq E''\}$. It is easy to see that $[r_K, r_L] = 1$ for $K \in \Sigma'$ and $L \in \Sigma''$. By Lemma 3.7, K and L commute for all $K \in \Sigma'$ and $L \in \Sigma''$, which contradicts Lemma 3.5. Hence W_0 is irreducible on E_q . \square

Lemma 3.13. *The group W_0 acts irreducibly on $T_p \otimes_{\mathbb{Z}_p} \mathbb{C}$.*

Proof. The proof is analogous to that of the previous lemma. \square

3.5. Root System. The aim of this subsection is to construct a root system on which W_0 acts as a crystallographic reflection group. The existence of such a root system is guaranteed by the following lemma.

Lemma 3.14. *There exists an irreducible root system on which W_0 acts as a crystallographic reflection group.*

Proof. Recall that $n \geq 3$. By Theorem 2.9, the quotient group W_0 is one of the crystallographic reflection groups $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$ and acts on the corresponding root system. \square

Let now $R = \{r_i \mid i \in I\}$ be a simple system of reflections in W_0 . We shall identify I with the set of nodes of the Dynkin diagram for W_0 . It is well known that every reflection in an irreducible reflection group W_0 is conjugate to a reflection in R .

Lemma 3.15. [4, Lemma 9.9] *Every reflection $r \in W_0$ has the form r_K for some SL_2 -subgroup $K \in \Sigma$.*

Proof. Let $r \in W_0$ be a reflection. Working our way back through the construction of the module T_p , one can easily see that the Prüfer p -rank of $[D, r]$ is 1.

Let $r_L \in W_0$ be a reflection which corresponds to a SL_2 -subgroup $L \in \Sigma$. By comparing the Prüfer p -ranks of the groups $C_D(r_L)$ and $C_D(r)$, we see that $Z = (C_H(r_L) \cap C_H(r))^\circ$ has Prüfer p -rank at least 1. Hence the subgroup $\langle L, H, r \rangle$ contains a non-trivial central p -torus; also note that $\langle L, H, r \rangle$ is a K -group, since D lies in H . It is well known that a finite irreducible reflection group contains at most two conjugate classes of reflections. Therefore, after replacing r and r_L by their appropriate conjugates in W_0 , we can assume without loss of generality that the images of r_L and r in W_0 correspond to adjacent nodes of the Dynkin diagram. Now we can easily see that $\langle L, H, r \rangle = Y * Z$ for some simple algebraic group Y of Lie rank 2, and that $r = r_K$ for some root SL_2 -subgroup K of Y such that $K \in \Sigma$. \square

3.6. Final Step. The next task is to prove that the conditions of Lyons's Theorem (Theorem 2.8) are satisfied.

Lemma 3.16. [4, Lemma 9.10] *Attach an SL_2 -subgroup $L_i \in \Sigma$ to each vertex of the Dynkin diagram I in such a way that the simple reflection r_i corresponding to this vertex is r_{L_i} in W_0 . Then the following statements hold.*

- (1) $[L_i, L_j] = 1$ if and only if $|r_i r_j| = 2$.
- (2) $\langle L_i, L_j \rangle$ is isomorphic to $(P)SL_3$ if and only if $|r_i r_j| = 3$.
- (3) $\langle L_i, L_j \rangle$ is isomorphic to $(P)Sp_4$ if and only if $|r_i r_j| = 4$.
- (4) L_i and L_j are embedded in $\langle L_i, L_j \rangle$ as root SL_2 -subgroups.

Proof. It is well known that for each $i, j \in I$, the order $|r_i r_j|$ takes the values 2, 3 or 4, in a Dynkin diagram of type $A_n, B_n, C_n, D_n, E_6, E_7, E_8$ or F_4 . By Lemma 3.3, L_i and L_j either commute or generate $(P)SL_3$, $(P)Sp_4$ or G_2 . However $\langle L_i, L_j \rangle \cong G_2$ is not possible in our case since $|r_i r_j| = 6$ does not occur in I .

The 'only if' parts of (1) and (2) are easy to see. In the case of part (3), that is when L_i and L_j generate $(P)Sp_4$, we have to show that $|r_i r_j| \neq 2$. To get a contradiction, assume that L_i and L_j generate $(P)Sp_4$ and $|r_i r_j| = 2$. But then L_i and L_j are both short root SL_2 -subgroups. Note that r_i and r_j are simple reflections, and it can be checked by inspection that one of them must be a long reflection. This proves the 'only if' part of (3). Now parts (1), (2) and (3) follow from Lemma 3.3 and the previous discussion. Part (4) is a direct consequence of Lemma 3.3.

Lemma 3.17. *Each subgroup in Σ is isomorphic to $(P)SL_2(\mathbb{F})$ for the same field \mathbb{F} .*

Proof. By Lemma 3.5 any two subgroups of Σ are connected by a sequence of edges. Note that each pair which is connected by a single edge generates a simple group of Lie rank 2 by Lemma 3.3(2), hence their underlying fields coincide. Thus the underlying fields of any two subgroups in Σ coincide.

Finally, we are in a position to apply Lyons's Theorem. Set G_0 to be the subgroup of G generated by the subgroups L_i for $i \in I$. By Lyons's Theorem, G_0 is a simple algebraic group over \mathbb{F} with the Dynkin diagram I . Its Weyl group, with respect to the torus T , is W_0 , hence G_0 contains all subgroups from Σ . Therefore by Lemma 3.4, $G_0 = G$ is a Chevalley group over \mathbb{F} . Since G is of p' -type, \mathbb{F} is of characteristic different from p . $\square \square$

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